Recall that a differential equation is an equation that involves $x$, $y$ and derivatives of $y$. A function $y = f(x)$ is called a solution of a differential equation if the equation is satisfied when $y$ and its derivatives are replaced by $f(x)$ and its derivatives. For example:

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>$y' + 2y = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Particular Solution</td>
<td>$f(x) = 5e^{-2x}$</td>
</tr>
<tr>
<td>Derivative of Solution</td>
<td>$f'(x) = -10e^{-2x}$</td>
</tr>
<tr>
<td>Verify</td>
<td>$-10e^{-2x} + 2(5e^{-2x}) = 0$</td>
</tr>
</tbody>
</table>

Differential equations have both general solutions and particular solutions. In the example above, $f(x) = 5e^{-2x}$ is a particular solution, while $f(x) = Ce^{-2x}$ would be a general solution.

1.1 Verify that $f(x) = Ce^{-2x}$ is a general solution for the differential equation $y' + 2y = 0$.

1.2 Determine whether each function is a solution to the differential equation $y'' - y = 0$.

   a) $y = \sin x$

   b) $y = 4e^{-x}$

   c) $y = Ce^x$
Recall that particular solutions of a differential equation are found from the general solution and an initial condition.

1.3 Determine the particular solution of the differential equation $y' = 3x^2$ that passes through the point (2, 4)


2 SLOPE FIELDS

Differential equations are often written in the form $y' = f(x,y)$, where $f(x,y)$ is an expression in terms of $x$ and $y$. This format allows the differential equation to be used to create a slope field.

2.1 Generate a slope field for the differential equation $y' = x - y$ by filling in the table and then marking each slope on the graph at right. Note that each dot represents one unit.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$y'$</th>
<th>$x$</th>
<th>$y$</th>
<th>$y'$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2.2 Generate a slope field for $\frac{dy}{dx} = -\frac{x}{y}$
2.3 Generate a slope field for \( \frac{dy}{dx} = 2y \)

2.4 Match each slope field with its differential equation.

- \( y' = x - 2 \)
- \( y' = x + y \)
- \( y' = y + \frac{1}{2} \)

2.5 Practice from AP Tests

The slope field for a differential equation is shown at the right. Which statement is true for solutions of the differential equation?

I. For \( x < 0 \) all solutions are decreasing.
II. All solutions level off near the \( x \)-axis.
III. For \( y > 0 \) all solutions are increasing.

(A) I only  (B) II only  (C) III only
(D) II and III only  (E) I, II, and III

The slope field for the differential equation \( \frac{dy}{dx} = \frac{x^2y + y^2}{4x + 2y} \) will have vertical segments when

(A) \( y = 2x \), only
(B) \( y = -2x \), only
(C) \( y = -x^2 \), only
(D) \( y = 0 \), only
(E) \( y = 0 \) or \( y = -x^2 \)
2.6 Practice Problems from Larson
Sketch the particular solution that passes through the indicated point, and use integration to find the equation of the particular solution

Larson Section 4.1

51. \( \frac{dy}{dx} = \frac{1}{2}x - 1, (4, 2) \)

Larson Section 4.5

54. \( \frac{dy}{dx} = -\frac{1}{x^2}, x > 0, (1, 3) \)

44. \( \frac{dy}{dx} = x^2(x^3 - 1)^2 \) 
(1, 0)
45. \( \frac{dy}{dx} = x \cos x^2 \)

\((0, 1)\)

Larson Section 5.2

49. \( \frac{dy}{dx} = \frac{1}{x + 2} \), \((0, 1)\)

50. \( \frac{dy}{dx} = \frac{\ln x}{x} \), \((1, -2)\)
131. \( \frac{dy}{dx} = 2e^{-x/2}, \quad (0, 1) \)

132. \( \frac{dy}{dx} = xe^{-0.2x^2}, \quad \left(0, -\frac{3}{2}\right) \)

The slope field from a certain differential equation is shown above. Which of the following could be a specific solution to that differential equation?

(A) \( y = x^2 \)  
(B) \( y = e^x \)  
(C) \( y = e^{-x} \)  
(D) \( y = \cos x \)  
(E) \( y = \ln x \)

Recommended homework Problems: 6.1: 53-54, 57-62
3 SOLVING DIFFERENTIAL EQUATIONS

So far you have solved differential equations in the form of $y' = f(x)$ by integrating. Now we will solve slightly more complicated differential equations. The calculus will be the same, but we will have to apply algebra before we can integrate. Before we can integrate, we must get all of the terms that include $y$ on one side of the equation, and all the terms that include $x$ on the other. This includes re-writing $y'$ as $\frac{dy}{dx}$. For example:

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>$y' = \frac{dy}{dx} = \frac{2x}{y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Separate Variables</td>
<td>$ydy = 2xdx$</td>
</tr>
<tr>
<td>Integrate</td>
<td>$\int y,dy = \int 2xdx$</td>
</tr>
<tr>
<td>Only one $+C$ needed</td>
<td>$\frac{1}{2}y^2 = \frac{2}{2}x^2 + C$</td>
</tr>
<tr>
<td>Simplify as needed</td>
<td>$y^2 = x^2 + C_1$</td>
</tr>
</tbody>
</table>

Note in the example above that the $C$ in the last equation is different than the $C$ in the previous equation, since it has been multiplied by 2 along with the rest of the equation. It is customary to indicate this change with a subscript. It is not necessary to write ‘$2C$’ in the last form, since $C$ can be any number.

3.1 Find the general solution of $(x^2 + 4) \frac{dy}{dx} = xy$.

3.2 Given the initial condition $y(1) = \frac{1}{25}$ find a particular solution for the differential equation $\frac{dy}{dx} = 6y^2x$. 
There is almost always a differential equation problem in the free-response section of the AP test. There are particular steps they look for and award points for. These steps usually are:

- Separation of Variables 1 point
- Integration of each side 1 or 2 points
- Includes +C 1 point (seriously. One point just for writing +C)
- Use Initial Conditions 1 point
- Solves into $y = f(x)$ form 1 point

This is 5 or 6 points out of 108 on the test. Practice following these steps on the following two problems.

3.4 AP Calculus AB 1998 FRQ #4: Let $f$ be a function with $f(1) = 4$ such that for all points $(x, y)$ on the graph of $f$ the slope is given by $\frac{3x^2 + 1}{2y}$.

a. Find the slope of the graph of $f$ at the point where $x = 1$.
b. Write an equation for the line tangent to the graph of $f$ at $x = 1$ and use it to approximate $f(1.2)$.
c. Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = \frac{3x^2 + 1}{2y}$ with the initial condition $f(1) = 4$.
d. Use your solution from part c to find $f(1.2)$.  


3.5 Solve the following differential equation and find the domain of the solution.

\[ y' = \frac{xy^3}{\sqrt{1+x^2}} \quad y(0) = -1 \]


4 EXponential GROWTH AND DECAY

A few months ago, 20 great horned sheep were released onto Mt. Lemmon, with the goal of establishing a stable population. The park rangers are always worried when they find another sheep killed by a mountain lion, because a dead sheep cannot breed and add to the population. The growth of a population is dependent on the size of the existing population. This is true in many real-life situations. The rate of change of a variable is proportional to the value of that variable. If \( y \) is a function of time \( t \), the proportion can be written as:

Let's solve this differential equation:

\[
\frac{dy}{dt} = ky, \text{ then } y = Ce^{kt}. \text{ } C \text{ is always the initial amount present/the } y\text{-intercept of the graph.}
\]
4.1 The rate of change of $y$ is proportional to $y$. When $t = 0$, $y = 2$, and when $t = 2$, $y = 4$. What is the value of $y$ when $t = 3$?

4.2 Suppose that 10 grams of the plutonium isotope $^{239}Pu$ was released in the Chernobyl nuclear accident. The half-life of Plutonium 239 is 24,100 years. How long will it take for the 10 grams to decay to 1 gram?

4.3 Suppose an experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?
Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between its temperature and the surrounding temperature. Let $y$ represent the temperature (in degrees F) of an object in a room whose temperature is kept at a constant 60°. If the object cools from 100° to 90° in 10 minutes, how much longer will it take for its temperature to decrease to 80°?

Recommended homework problems: 6.2: 11-13, 15, 16, 33-37, 43-45

5 DETECTIVE PROJECT

A detective is investigating a man’s death. The man was seen falling from the roof of a building, hitting the ground after exactly three seconds. The suspect, who was on the roof with the man, claims that the man jumped, but the detective suspects foul play. Assume that acceleration due to gravity is -32 feet per second per second.

5.1 Assume the man accidently fell. Write a differential equation with initial conditions to model that situation. Solve the differential equation to find the velocity function for the fall. Determine the position function for the fall, and use this to determine how tall the building must be.
5.2 Assume the man jumped with an initial velocity of 2 feet per second. Proceed as above to find out how tall the building must be.

5.3 Now assume the man was pushed with an initial velocity of -3 feet per second. Again, determine how tall the building must be.

5.4 What additional information is needed to determine whether foul play was involved? Explain.

5.5 The detective discovered that the building is 20 stories tall and each story is about 8 feet. What should the detective conclude?